Polynomial Optimization in Quantum Information Theory

Sabine Burgdorf

University of Konstanz

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Real Algebraic Geometry and Optimization
Warm Up

- **Entanglement** is one of the key features in Quantum Information
- Bell ’64:

- How to distinguish $\mathcal{C}$ and $\mathcal{Q}$?
- What is the correct definition for $\mathcal{Q}$? Does it matter?
- Can Polynomial Optimization help to understand these sets?
RAG and POP basics

Polynomial Optimization

- $f \in \mathbb{R}[X]$ polynomial in commuting variables
- $g_0 = 1, g_1, \ldots, g_r \in \mathbb{R}[X]$ defining a semi-algebraic set:

$$K = \{a \in \mathbb{R}^n \mid g_0(a) \geq 0, \ldots, g_r(a) \geq 0\}$$

- Want to minimize $f$ over $K$

$$f_* = \inf_{a \in K} f(a) \quad \text{s.t. } a \in K$$

$$= \sup_{a \in \mathbb{R}} \quad \text{s.t. } f - a \geq 0 \text{ on } K$$

- NP-hard
RAG and POP basics

RAG helps

\[ f_\star = \sup a \in \mathbb{R} \quad \text{s.t.} \quad f - a \geq 0 \text{ on } K \]

NP-hard 😞

- \[ M(g) := \{ p = \sum_j h_j^2 g_{ij} \text{ for some } h_i \in \mathbb{R}[X] \} \]
- sos relaxation

\[ f_{sos} = \sup a \in \mathbb{R} \quad \text{s.t.} \quad f - a \in M(g) \]

"SDP" 😊
RAG and POP basics

RAG helps

\[
f_\star = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \geq 0 \text{ on } K
\]

NP-hard 😞

- \(M(g) := \{ p = \sum_j h_j^2 g_{ij} \text{ for some } h_i \in \mathbb{R}[X] \}\)
- sos relaxation

\[
f_{\text{sos}} = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \in M(g)
\]

"SDP" 😊

- \(f_{\text{sos}}\) is always a lower bound but might be strict

- If \(M(g)\) is archimedean:
  \[f_\star = f_{\text{sos}}\]

\[x_1^4 x_2^2 + x_1^2 x_2^4 - 3 x_1^2 x_2^2 + 1\]
RAG and POP basics

SOS hierarchy

- $M(g)_t := \{p = \sum_j h_j^2 g_{ij} \text{ for some } h_i \in \mathbb{R}[X]_t\}$
- sos hierarchy

$$f_t = \sup a \in \mathbb{R} \text{ s.t. } f - a \in M(g)_t$$

- We have
  - $f_t \leq f_{t+1} \leq f_*$
  - $f_t$ converges to $f_{sos}$ as $t \to \infty$
  - If $M(g)$ is archimedean: $f_{sos} = f_*$
RAG and POP basics

SOS hierarchy

- \( M(g)_t := \{ p = \sum_j h_j^2 g_{ij} \text{ for some } h_i \in \mathbb{R}[X]_t \} \)
- sos hierarchy
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  f_t = \sup a \in \mathbb{R} \text{ s.t. } f - a \in M(g)_t
  \]

- We have
  - \( f_t \leq f_{t+1} \leq f_* \)
  - \( f_t \) converges to \( f_{sos} \) as \( t \to \infty \)
  - If \( M(g) \) is archimedean: \( f_{sos} = f_* \)
- Certificate of exactness:
  - Flatness of dual solution
  - Allows extraction of optimizers
NC-RAG and NC-POP

NC Polynomials

▶ Want to replace scalar variables by matrices/operators
▶ Free algebra $\mathbb{R}\langle X \rangle$ with noncommuting variables $X_1, \ldots, X_n$
▶ Polynomial

\[ f = \sum_{w} f_w w \]

▶ Let $A \in (S^d)^n$: $f(A) = f_1 I_d + f_{X_1} A_1 + f_{X_2 X_1} A_2 A_1 \ldots$
NC-RAG and NC-POP

NC Polynomials

- Want to replace scalar variables by matrices/operators
- Free algebra $\mathbb{R}\langle X \rangle$ with noncommuting variables $X_1, \ldots, X_n$
- Polynomial
  \[ f = \sum_w f_w w \]
- Let $A \in (S^d)^n$: $f(A) = f_1 I_d + f_{X_1} A_1 + f_{X_2 X_1} A_2 A_1 \ldots$
- Add involution $\ast$ on $\mathbb{R}\langle X \rangle$
  - fixes $\mathbb{R}$ and $\{X_1, \ldots, X_n\}$ pointwise
  - $X_i^\ast = X_i$
- Consequence
  \[ f^\ast f(A) = f(A)^T f(A) \succeq 0 \]
NC-RAG and NC-POP

NC Polynomial Optimization

- Let $f \in \mathbb{R}\langle X\rangle$
- $g_0 = 1$, $g_1, \ldots, g_r \in \mathbb{R}\langle X\rangle$ defining a semi-algebraic set:

  $$K = \{A \mid g_0(A) \succeq 0, \ldots, g_r(A) \succeq 0\}$$

- Want to minimize $f$ over $K$

$$f^* = \sup a \in \mathbb{R} \text{ s.t. } f - a \succeq 0 \text{ on } K$$
NC-RAG and NC-POP

Eigenvalue optimization

- Let $f \in \mathbb{R}\langle X \rangle$

$$f_{nc} = \sup a \in \mathbb{R} \quad \text{s.t.} \quad f - a \succeq 0 \text{ on } K$$

NP-hard 😞

- Observation: Checking if $f = \sum_i h_i^* h_i$ is an SDP
  so as well checking $f = \sum_j h_j^* g_{ij} h_j$ (with degree bounds)
NC-RAG and NC-POP

Eigenvalue optimization

- Let $f \in \mathbb{R}\langle X \rangle$

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- Observation: Checking if $f = \sum_i h_i^* h_i$ is an SDP so as well checking $f = \sum_j h_j^* g_i h_j$ (with degree bounds)

- sos relaxation

\[ M_{nc}(g) := \{ p = \sum_j h_j^* g_i h_j \text{ for some } h_i \in \mathbb{R}\langle X \rangle \} \]

\[ f_{sos} = \sup a \in \mathbb{R} \text{ s.t. } f - a \in M_{nc}(g) \]

- Fact: $f_{sos} \leq f_{nc}$

- Theorem (Helton et al.): If $M_{nc}(g)$ is archimedean, then $f_{sos} = f_{nc}$. 
NC-RAG and NC-POP

Eigenvalue optimization

- Let \( f \in \mathbb{R}\langle X \rangle \)
  \[
  f_{nc} = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \succeq 0 \text{ on } K
  \]
  NP-hard 😞

- \( M_{nc}(g)_t := \{ p = \sum_j h_j^* g_j h_j \text{ for some } h_j \in \mathbb{R}\langle X \rangle_t \} \)

- sos hierarchy
  \[
  f_t = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \in M_{nc}(g)_t
  \]
  SDP 😊

- \( f_t \leq f_{t+1} \leq f_{nc} \) but inequalities might be strict

- \( f_t \) converges to \( f_{sos} \) as \( t \to \infty \)

- If \( M_{nc}(g) \) is archimedean: \( f_{sos} = f_{nc} \) and hence \( f_t \to f_{nc} \) as \( t \to \infty \)
NC-RAG and NC-POP

Trace optimization

- Let \( f \in \mathbb{R}\langle X\rangle \)

\[
f_{tr} = \sup a \in \mathbb{R} \quad \text{s.t.} \quad \text{Tr}(f - a) \geq 0 \text{ on } K \quad \text{NP-hard}
\]

- \( K \) contains only operators, for which a trace is defined
NC-RAG and NC-POP

Trace optimization

- Let $f \in \mathbb{R}\langle X \rangle$

\[
f_{tr} = \sup a \in \mathbb{R} \quad \text{s.t.} \quad \text{Tr}(f - a) \geq 0 \text{ on } K
\]

NP-hard 😞

- $K$ contains only operators, for which a trace is defined
- If $f = \sum_j h_j^* g_j h_j + \sum_k [p_k, q_k]$ then Tr($f(A)$) $\geq 0$ for all $A \in K$
- sos relaxation

\[
M_{tr}(g) := \{\sum_j h_j^* g_j h_j \text{ for some } h_i \in \mathbb{R}\langle X \rangle\} + [\mathbb{R}\langle X \rangle, \mathbb{R}\langle X \rangle]
\]

\[
f_{sos} = \sup a \in \mathbb{R} \quad \text{s.t.} \quad f - a \in M_{tr}(g)
\]

- Fact: $f_{sos} \leq f_{tr}$
- Theorem (B., Klep et al.): If $M_{tr}(g)$ is archimedean, then $f_{sos} = f_{tr}$. 
NC-RAG and NC-POP

Trace optimization

- Let $f \in \mathbb{R} \langle X \rangle$

$$f_{tr} = \sup a \in \mathbb{R} \quad \text{s.t.} \quad \text{Tr}(f - a) \geq 0 \text{ on } K$$

- $M_{tr}(g)_t := \{ \sum_j h_j^* g_i h_j \text{ for some } h_j \in \mathbb{R} \langle X \rangle_t \} + \sum [\mathbb{R} \langle X \rangle, \mathbb{R} \langle X \rangle]$ (NP-hard 😞)

- sos hierarchy

$$f_t = \sup a \in \mathbb{R} \quad \text{s.t.} \quad f - a \in M_{tr}(g)_t$$ (SDP 😊)

- $f_t \leq f_{t+1} \leq f_{tr}$ but inequalities might be strict

- $f_t$ converges to $f_{sos}$ as $t \to \infty$

- If $M_{tr}(g)$ is archimedean: $f_{sos} = f_{tr}$ and hence $f_t \to f_{tr}$ as $t \to \infty$
Back to Quantum Information

- **Entanglement** is one of the key features in Quantum Information
- Bell ’64:

![Diagram]

- How to distinguish $\mathcal{C}$ and $\mathcal{Q}$?
- What is the correct definition for $\mathcal{Q}$? Does it matter?
- Can Polynomial Optimization help to understand these sets?
Basics of quantum theory

- A quantum system corresponds to a Hilbert space $\mathcal{H}$
- Its states are unit vectors on $\mathcal{H}$
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- A state on a composite system is a unit vector $\psi$ on a tensor Hilbert space, e.g. $\mathcal{H}_A \otimes \mathcal{H}_B$
- $\psi$ is entangled if it is not a product state $\psi_A \otimes \psi_B$ with $\psi_A \in \mathcal{H}_A, \psi_B \in \mathcal{H}_B$
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- A state $\psi \in \mathcal{H}$ can be measured
  - outcomes $a \in A$
  - POVM: a family $\{E_a\}_{a \in A} \subseteq B(\mathcal{H})$ with $E_a \geq 0$ and $\sum_{a \in A} E_a = 1$
  - probability of getting outcome $a$ is $p(a) = \psi^T E_a \psi$. 
Nonlocal bipartite correlations

- Question sets $S$, $T$, Answer sets $A$, $B$
- No (classical) communication

- Which correlations $p(a, b \mid s, t)$ are possible?
Correlations

Classical strategy $\mathcal{C}$

Independent probability distributions $\{p^a_s\}_a$ and $\{p^b_t\}_b$:

$$p(a, b \mid s, t) = p^a_s \cdot p^b_t$$

shared randomness: allow convex combinations

Quantum strategy $\mathcal{Q}$

POVMs $\{E^a_s\}_a$ and $\{F^b_t\}_b$ on Hilbert spaces $H_A, H_B$,

$$\psi \in H_A \otimes H_B$$:

$$p(a, b \mid s, t) = \psi^T (E^a_s \otimes F^b_t) \psi$$

▶ Nonlocality:

$$(E^a_s \otimes 1)(1 \otimes F^b_t) = (1 \otimes F^b_t)(E^a_s \otimes 1)$$

▶ If $\psi = \psi_A \otimes \psi_B$ then we have classical correlation
Correlations

Classical strategy $\mathcal{C}$

Independent probability distributions $\{p^a_s\}_a$ and $\{p^b_t\}_b$:

$$p(a, b \mid s, t) = p^a_s \cdot p^b_t$$

shared randomness: allow convex combinations

Quantum strategy $\mathcal{Q}$

POVMs $\{E^a_s\}_a$ and $\{F^b_t\}_b$ on Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$:

$$p(a, b \mid s, t) = \psi^T (E^a_s \otimes F^b_t) \psi$$

- Nonlocality: $(E^a_s \otimes 1)(1 \otimes F^b_t) = (1 \otimes F^b_t)(E^a_s \otimes 1)$
- If $\psi = \psi_A \otimes \psi_B$ then we have classical correlation
More correlations

Quantum strategy $Q$

POVMs $\{E^a_s\}_a$ and $\{F^b_t\}_b$ on Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$:

$$p(a, b \mid s, t) = \psi^T (E^a_s \otimes F^b_t) \psi$$
More correlations

Quantum strategy $Q$

POVMs $\{E^a_s\}_a$ and $\{F^b_t\}_b$ on Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$:

$$p(a, b \mid s, t) = \psi^T (E^a_s \otimes F^b_t) \psi$$

Quantum strategy $Q_c$

POVMs $\{E^a_s\}_a$ and $\{F^b_t\}_b$ on a joint Hilbert space, but $[E^a_x, F^b_y] = 0$:

$$p(a, b \mid s, t) = \psi^T (E^a_s \cdot F^b_t) \psi$$

Fact

$$\mathcal{C} \subseteq Q \subseteq \overline{Q} \subseteq Q_c$$
Tsirelson’s problem

Fact

\[ \mathcal{C} \subseteq \mathcal{Q} \subseteq \overline{\mathcal{Q}} \subseteq \mathcal{Q}_c \]

- Bell: \( \mathcal{C} \neq \mathcal{Q} \)
- closure conjecture [Slofstra ’16]: \( \mathcal{Q} \neq \overline{\mathcal{Q}} \)
- weak Tsirelson [Slofstra ’16]: \( \mathcal{Q} \neq \mathcal{Q}_c \)
- Dykema et al. ’17: Concrete example in a decent subset of \( \mathcal{Q} \)
- strong Tsirelson (open): Is \( \overline{\mathcal{Q}} = \mathcal{Q}_c \) ?
- strong Tsirelson is equivalent to Connes embedding problem
Nonlocal games

- Characterized by
  - 2 sets of questions $S, T$, asked with probability distribution $\pi$
  - 2 sets of answers $A, B$
  - A winning predicate $V : A \times B \times S \times T \rightarrow \{0, 1\}$
Nonlocal games

- Characterized by
  - 2 sets of questions $S, T$, asked with probability distribution $\pi$
  - 2 sets of answers $A, B$
  - A winning predicate $V : A \times B \times S \times T \rightarrow \{0, 1\}$

- Winning probability (value of the game)

$$\omega = \sup_{p} \sum_{s \in S, t \in T} \pi(s, t) \sum_{a \in A, b \in B} V(a, b; s, t) p(a, b | s, t)$$

$$= \sup_{p} \sum_{a, b, s, t} f_{abst} p(a, b | s, t)$$

- Optimize over correlations $p \in \{C, Q, Q_c\}$
SOS relaxation over $\mathcal{C}$

$$\omega_{\mathcal{C}} = \sup_p \sum_{a,b,s,t} f_{abst} p_s^a \cdot p_t^b$$
SOS relaxation over $\mathcal{C}$

\[
\omega_{\mathcal{C}} = \sup_{p} \sum_{a,b,s,t} f_{\text{abst}} p_s^a \cdot p_t^b
\]

- We can write this as POP:
  - \( f((p, q)) := \sum_{a,b,s,t} f_{\text{abst}} p_s^a \cdot q_t^b \in \mathbb{R}[p, q] \)
  - \( K = \{(p, q) \mid p_s^a, q_t^b \geq 0, \sum_a p_s^a = \sum_b q_t^b = 1\} \)
  - \( M(g) \) is archimedean
SOS relaxation over $C$

$$\omega_C = \sup_p \sum_{a,b,s,t} f_{abst} p_s^a \cdot p_t^b$$

- We can write this as POP:
  - $f((p, q)) := \sum_{a,b,s,t} f_{abst} p_s^a \cdot q_t^b \in \mathbb{R}[p, q]$
  - $K = \{(p, q) \mid p_s^a, q_t^b \geq 0, \sum_a p_s^a = \sum_b q_t^b = 1\}$
  - $M(g)$ is archimedean

- Hence

$$\begin{align*}
\omega_C &= \sup f(p, q); \quad \text{s.t. } (p, q) \in K \\
&= \inf a \in \mathbb{R}; \quad \text{s.t. } a - f \geq 0 \text{ on } K \\
&= \inf a \in \mathbb{R}; \quad \text{s.t. } a - f \in M(g) \quad (f_{sos}) \\
&\leq \inf a \in \mathbb{R}; \quad \text{s.t. } a - f \in M(g)_t \quad (f_t)
\end{align*}$$

- Converging hierarchy of SDP upper bounds
SOS relaxation over $Q_c$

\[ \omega_{Q_c} = \sup \sum_{a,b,s,t} f_{abst} \psi^T (E_s^a \cdot F_t^b) \psi \]
SOS relaxation over \( Q_c \)

\[
\omega_{Q_c} = \sup \sum_{a,b,s,t} f_{abst} \psi^T (E^a_s \cdot F^b_t) \psi
\]

- We can write this as NC-POP:
  - \( f(E, F) := \sum_{a,b,s,t} f_{abst} E^a_s \cdot F^b_t \in \mathbb{R} \langle E, F \rangle \)
  - \( K = \{(E, F) \mid E_s, F_t \succeq 0, \sum_a E^a_s = \sum_b F^b_t = 1, [E^a_s, F^b_t] = 0\} \)
  - \( M_{nc}(g) \) is archimedean
SOS relaxation over $Q_c$

\[
\omega_{Q_c} = \sup \sum_{a,b,s,t} f_{abst} \psi^T (E^a_s \cdot F^b_t) \psi
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  - \( M_{nc}(g) \) is archimedean

- Hence

\[
\omega_C = \sup \psi^T f(E, F) \psi; \quad \text{s.t.} \quad (E, F) \in K
\]

\[
= \inf a \in \mathbb{R} \quad \text{s.t.} \quad a - f \succeq 0 \text{ on } K
\]

\[
= \inf a \in \mathbb{R} \quad \text{s.t.} \quad a - f \in M_{nc}(g) \quad (f_{sos})
\]

\[
\leq \inf a \in \mathbb{R} \quad \text{s.t.} \quad a - f \in M_{nc}(g)_t \quad (f_t)
\]

- Converging hierarchy of SDP upper bounds
SOS relaxation over $Q$

$$\omega_Q = \sup \sum_{a,b,s,t} f_{abst} \Tr(E^a_s \otimes F^b_t)$$

- Cameron et al.: For most games we have $p(a, b \mid s, t) = \Tr(\tilde{E}^a_s \tilde{F}^b_t)$ with $\tilde{E}^a_s, \tilde{F}^b_t \succeq 0$, $\sum_a \tilde{E}^a_s = \sum_b \tilde{F}^b_t = D$ with $\Tr(D^2) = 1$
SOS relaxation over $\mathcal{Q}$

$$\omega_{\mathcal{Q}} = \sup \sum_{a,b,s,t} f_{\text{abst}} \text{Tr}(E^a_s \otimes F^b_t)$$

- Cameron et al.: For most games we have $p(a, b | s, t) = \text{Tr}(\tilde{E}^a_s \tilde{F}^b_t)$ with $\tilde{E}^a_s, \tilde{F}^b_t \succeq 0$, $\sum_a \tilde{E}^a_s = \sum_b \tilde{F}^b_t = D$ with $\text{Tr}(D^2) = 1$

- We can write this as NC-POP:
  - $f(E, F) := \sum_{a,b,s,t} f_{\text{abst}} E^a_s \cdot F^b_t \in \mathbb{R} \langle E, F \rangle$
  - $K = \{(E, F) | E_s, F_t \succeq 0, \sum_a E^a_s = \sum_b F^b_t = D, \text{Tr}(D^2) = 1\}$

- Hence

$$\omega_C = \sup \text{Tr} f(E, F); \quad \text{s.t.} \quad (E, F, D) \in K \leq \inf a \in \mathbb{R} \quad \text{s.t.} \quad a - f \in M_{\text{tr}}(g)$$

- Converging sequence of upper SDP bounds
CHSH Game

- Questions $S = T = \{0, 1\}$, Answers $A = B = \{0, 1\}$
- Alice & Bob win, if $a + b \equiv st \mod 2$
CHSH Game

- Questions \( S = T = \{0, 1\} \), Answers \( A = B = \{0, 1\} \)
- Alice & Bob win, if \( a + b \equiv st \ mod \ 2 \)
- \( \omega_C = \frac{3}{4} \)
- \( \omega_Q = \omega_{Q_c} = \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.854 \)
- 1\textsuperscript{st} level of SOS hierarchies are exact
CHSH Game

- Questions $S = T = \{0, 1\}$, Answers $A = B = \{0, 1\}$
- Alice & Bob win, if $a + b \equiv st \mod 2$
- $\omega_c = \frac{3}{4}$
- $\omega_Q = \omega_{Q_c} = \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.854$
- 1st level of SOS hierarchies are exact

- Alternative formulation:
- 2 measurements with 2 outcomes each: $E_s^0, E_s^1, F_t^0, F_t^1$
- Setting $E_s := E_s^0 - E_s^1$, $F_t := F_t^0 - F_t^1$ one obtains the CHSH inequality

$$f_{CHSH} := E_0 F_0 + E_0 F_1 + E_1 F_0 - E_1 F_1$$

- Optimizing $f_{CHSH}$ over variants of $C, Q$ give $\omega_C, \omega_Q$
\( l_{3322} \) inequality

- Questions \( S = T = \{0, 1, 2\} \), Answers \( A = B = \{0, 1\} \)

\[
f := E_0F_0 + E_0F_1 + E_0F_2 + E_1F_0 + E_1F_1 - E_1F_3 + E_2F_0 - E_2F_1 - E_0 - 2F_0 - F_1
\]
Questions $S = T = \{0, 1, 2\}$, Answers $A = B = \{0, 1\}$

\[
f := E_0 F_0 + E_0 F_1 + E_0 F_2 + E_1 F_0 + E_1 F_1 - E_1 F_3 + E_2 F_0 - E_2 F_1
- E_0 - 2F_0 - F_1
\]

Maximizing over $C$: $f_* \leq 0$

Best lower bound: 0.250875384
$l_{3322}$ inequality

- Questions $S = T = \{0, 1, 2\}$, Answers $A = B = \{0, 1\}$

$$f := E_0 F_0 + E_0 F_1 + E_0 F_2 + E_1 F_0 + E_1 F_1 - E_1 F_3 + E_2 F_0 - E_2 F_1 - E_0 - 2F_0 - F_1$$

- Maximizing over $C$: $f_\ast \leq 0$
- Best lower bound: 0.250875384

- NC-SOS upper bounds:

<table>
<thead>
<tr>
<th>level</th>
<th>psd</th>
<th>trace</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.375</td>
<td>0.375</td>
</tr>
<tr>
<td>2</td>
<td>0.25094006</td>
<td>0.2509397</td>
</tr>
<tr>
<td>3</td>
<td>0.25087556</td>
<td>0.2508754</td>
</tr>
</tbody>
</table>

- Pal & Vertesi computed (eigenvalue) SOS-bounds for 240 Bell inequalities of which 20 are not matching ($\geq 10^{-4}$) the lower bound. 4 of them get exact ($\leq 10^{-8}$) using trace SOS-bounds, about 1/2 of them improve
Quantum coloring as feasibility problem

\[ \sum_{i \in [t]} x_i u = 1 \quad \forall u \in V(G), \forall i \neq j, \forall u \in V(G), \forall uv \in E(G) = \min_{t \in \mathbb{N}} \text{s.t.} \]

We can write this as

\[ \min_{t \in \mathbb{N}} \text{s.t.} \exists \text{operator solution of } (\ast) \]
Quantum coloring as feasibility problem

\[ \chi(G) = \min t \in \mathbb{N} \text{ s.t. } x_i^u \in \{0, 1\}, u \in V(G), \; i \in [t], \]

\[ \sum_{i \in [t]} x_i^u = 1 \quad \forall u \in V(G), \]

\[ x_i^i x_j^j = 0 \quad \forall i \neq j, \forall u \in V(G), \]

\[ x_i^u x_j^v = 0 \quad \forall uv \in E(G) \]
Quantum coloring as feasibility problem

\[ \chi_q(G) = \min t \in \mathbb{N} \text{ s.t. } x_i^u \succeq 0, u \in V(G), i \in [t], \]

\[ \sum_{i \in [t]} x_i^u = 1 \quad \forall u \in V(G), \]

\[ x_i^u x_j^u = 0 \quad \forall i \neq j, \forall u \in V(G), \quad (\ast) \]

\[ x_i^u x_i^v = 0 \quad \forall uv \in E(G) \]

\[ (x_i^u)^2 = x_i^u \quad \forall u \in V(G), i \in [t] \]
Quantum coloring as feasibility problem

\[ \chi_q(G) = \min t \in \mathbb{N} \text{ s.t. } x_{ui}^i \geq 0, u \in V(G), i \in [t], \]

\[ \sum_{i \in [t]} x_{ui}^i = 1 \quad \forall u \in V(G), \]

\[ x_{ui}^i x_{uj}^j = 0 \quad \forall i \neq j, \forall u \in V(G), \quad (\ast) \]

\[ x_{ui}^i x_{uv}^j = 0 \quad \forall uv \in E(G) \]

\[ (x_{ui}^i)^2 = x_{ui}^i \quad \forall u \in V(G), i \in [t] \]

- We can write this as

\[ \min t \in \mathbb{N} \text{ s.t. } \exists \text{ operator solution of } (\ast) \]
Nullstellensätze

Let $g_1, \ldots, g_r \in \mathbb{C}[X]$

Theorem (weak Nullstellensatz)

Let $I = (g_1, \ldots, g_r)$, $V(I) := \{a \in \mathbb{C}^n \mid g_1(a) = \cdots = g_r(a) = 0\}$. Then

$$V(I) = \emptyset \iff 1 \in I.$$
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Let $g_1, \ldots, g_r \in \mathbb{C}\langle X \rangle$

**Theorem (Amitsur Nullstellensatz)**

Let $Z(I) := \{A \in R^n \mid R \text{ primitive ring}, g_1(A) = \cdots = g_r(A) = 0\}$. Then

$$Z(I) = \emptyset \iff 1 \in (g_1, \ldots, g_r).$$

- We have an algorithm to compute NC Gröbner bases, but it might not terminate...
Against all odds...\(^1\)

- Gröbner basis: \(4 \leq \chi_q(G_{13})\)

\(^1\)with Piovesan, Mancinska, Roberson
Against all odds...\(^1\)

- Gröbner basis: \(4 \leq \chi_q(G_{13}) \leq \chi(G_{13}) = 4\)
- Consequence \(\chi_q(G_{14}) = 4 < 5 = \chi(G_{14})\)

\(^1\)with Piovesan, Mancinska, Roberson
Final Remarks

- Quantum theory gives archimedean property for NC-SOS relaxations
- Dual side (linear forms & moments) offers even more bounds (Laurent et al.)
- We can transfer the flatness machinery & might obtain concrete optimizer/strategies
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Open problems

- What is the geometry of (quantum) correlations?
- Is there always a finite dimensional solution/strategy for a finite game?
- How can we detect optimality if there is no finite dimensional solution?
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Thank you for your attention.
POEMA
Polynomial Optimization, Efficiency through Moments and Algebra
Marie Skłodowska-Curie Innovative Training Network
2019-2022

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